Various roots can be related to solutions of equations, & we want such numbers to be in simplified form for easier calculations & algebraic manipulations. We need the two properties of radicals stated here for square roots.

Properties of Square Roots

If a & b are positive real numbers then

1 $\sqrt{ab} = \sqrt{a}\sqrt{b}$

2 $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

A square root is considered to be in simplest form when the radicand has no perfect square as a factor.

The number 200 is not a perfect square & using a calculator we will find $\sqrt{200} \approx 14.1421$. Now to simplify, we can use property 1 of square roots & any of the following three approaches.

1. Factor 200 as 4 x 50 because 4 is a perfect square. This gives

$$\sqrt{200} = \sqrt{4 \cdot 50} = \sqrt{4} \cdot \sqrt{50} = 2\sqrt{50}$$

However, $2\sqrt{50}$ Is not in its simplest form because 50 has a perfect square factor, 25. Thus to complete the process we have

$$\sqrt{200} = 2\sqrt{50} = 2\sqrt{25 \cdot 2} = 2\sqrt{25} \cdot \sqrt{2} = 2 \cdot 5 \cdot \sqrt{2} = 10\sqrt{2}$$

2. Note that 100 is a perfect square factor of 200 & 200 = 100 x 2.

$$\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$$

3. Use prime factors.

$$\begin{aligned} \sqrt{200} &= \sqrt{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} \\ &= \sqrt{2 \cdot 2 \cdot 5 \cdot 5} \cdot \sqrt{2} \\ &= \sqrt{2 \cdot 2} \cdot \sqrt{5 \cdot 5} \cdot \sqrt{2} \\ &= 2 \cdot 5 \cdot \sqrt{2} \\ &= 10\sqrt{2} \end{aligned}$$

Of these three approaches, the second appears to be the easiest because it has the fewest steps. However, "seeing" the largest perfect square factor may be difficult. If you do not immediately see a perfect square factor, proceed by finding other factors/prime factors as illustrated.

A. $\sqrt{48}$

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$$

16 is the largest perfect square factor.

B. $\sqrt{63}$

$$\sqrt{63} = \sqrt{9 \cdot 7} = \sqrt{9} \cdot \sqrt{7} = 3\sqrt{7}$$

9 is the largest perfect square factor.

C. $\sqrt{\frac{75}{16}}$

$$\sqrt{\frac{75}{16}} = \frac{\sqrt{75}}{\sqrt{16}} = \frac{\sqrt{25 \cdot 3}}{\sqrt{16}} = \frac{\sqrt{25} \cdot \sqrt{3}}{\sqrt{16}} = \frac{5\sqrt{3}}{4}$$

To simplify square root expressions that contain variables, such as $\sqrt{x^2}$, we must be aware of whether the variable represents a positive real number ($x > 0$), zero ($x = 0$), or a negative number ($x < 0$). For example:

If $x = 0$, then $\sqrt{x^2} = \sqrt{0^2} = \sqrt{0} = 0 = x$

If $x = 5$, then $\sqrt{x^2} = \sqrt{5^2} = \sqrt{25} = 5 = x$

But, if $x = -5$ then $\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 \neq x$

In fact, if $x = -5$ then $\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 = |-5| = |x|$

Thus simplifying radical expressions w variables involves more detailed analysis than simplifying radical expressions w only constants.

Although using the absolute value when simplifying square roots is correct mathematically, we can avoid some confusion by assuming that the variable under the radical sign represents only positive real numbers/0. This eliminates the need for absolute value signs.

Therefore, we will often assume that $x > 0$ & write $\sqrt{x^2} = x$ in square root expressions.

a. $\sqrt{16y^2}$

$$\sqrt{16y^2} = 4y$$

b. $\sqrt{72a^2}$

$$\sqrt{72a^2} = \sqrt{36a^2} \cdot \sqrt{2} = 6a\sqrt{2}$$

c. $\sqrt{12x^2y^2}$

$$\sqrt{12x^2y^2} = \sqrt{4x^2y^2} = \sqrt{3} = 2xy\sqrt{3}$$

To find the square root of an expression w even exponents, divide the exponents by 2. For example,

$$x^2 \cdot x^2 = x^4 \longrightarrow \sqrt{x^4} = x^2$$
$$a^3 \cdot a^3 = a^6 \longrightarrow \sqrt{a^6} = a^3$$
$$\& \; y^5 \cdot y^5 = y^{10} \longrightarrow \sqrt{y^{10}} = y^5$$

To find the square root of an expression w odd exponents, factor the expression into two terms, one w exponent 1 & the other w an even exponent. For example,

$$x^3 = x^2 \cdot x \;\; \& \;\; y^9 = y^8 \cdot y$$

$$\sqrt{x^3} = \sqrt{x^2 \cdot x} = \sqrt{x^2} \cdot \sqrt{x} = x \cdot \sqrt{x} \;\; \& \;\; \sqrt{y^9} = \sqrt{y^8 \cdot y} = \sqrt{y^8} \cdot \sqrt{y} = y^4\sqrt{y}$$

a. $\sqrt{81x^4}$

$$\sqrt{81x^4} = 9x^2$$

The exponent 4 is divided by 2.

b. $\sqrt{64x^5y}$

$$\sqrt{64x^5y} = \sqrt{64x^4} \cdot \sqrt{xy} = 8x^2\sqrt{xy}$$

c. $\sqrt{18a^4b^6}$

$$\sqrt{18a^4b^6} = \sqrt{9a^4b^6} \cdot \sqrt{2} = 3a^2b^3\sqrt{2}$$

Each exponent is divided by 2.

d. $\sqrt{\frac{9a^{13}}{b^4}}$

$$\sqrt{\frac{9a^{13}}{b^4}} = \frac{\sqrt{9a^{13}}}{\sqrt{b^4}} = \frac{\sqrt{9a^{12}} \cdot \sqrt{a}}{\sqrt{b^4}} = \frac{3a^6\sqrt{a}}{b^2}$$

Recall $a, b > 0$.

When simplifying expressions w cube roots, we need to be aware of perfect cube numbers & variables w exponents that are multiples of 3. Thus exponents are divided by 3 in simplifying cube root expressions. For example,

$$x^2 \cdot x^2 \cdot x^2 = x^6 \longrightarrow \sqrt[3]{x^6} = x^2$$
$$a^3 \cdot a^3 \cdot a^3 = a^9 \longrightarrow \sqrt[3]{a^9} = a^3$$
$$\& \; y^5 \cdot y^5 \cdot y^5 = y^{15} \longrightarrow \sqrt[3]{y^{15}} = y^5$$

A cube root is considered to be in simplest form when the radicand has no perfect cube as a factor.

When finding cube roots, we need not be concerned about positive & negative values because cube roots of negative numbers are defined to be negative. For example:

We have $(-2)^3 = -8$ therefore $\sqrt[3]{-8} = -2$
Similarly, $(-5)^3 = -125$ therefore $\sqrt[3]{-125} = -5$

Thus $\sqrt[3]{x^3} = x$ whether $x \geq 0$ or $x < 0$.

a. $\sqrt[3]{54x^6}$

$$\sqrt[3]{54x^6} = \sqrt[3]{27x^6} \cdot \sqrt[3]{2} = 3x^2\sqrt[3]{2}$$

Note: 27 is a perfect cute & the exponent 6 is divisible by 3.

b. $\sqrt[3]{-40x^4y^{13}}$

$$\sqrt[3]{-40x^4y^{13}} = \sqrt[3]{-8x^3y^{12}} \cdot \sqrt[3]{5xy} = -2xy^4\sqrt[3]{5xy}$$

Note: –8 is a perfect cube & the exponents on the variables are separated so that one exponent on each variable is divisible but 3.

c. $\sqrt[3]{250a^8b^{11}}$

$$\sqrt[3]{250a^8b^{11}} = \sqrt[3]{125a^6b^9} \cdot \sqrt[3]{2a^2b^2} = 5a^2b^3\sqrt[3]{2a^2b^2}$$

Note: 125 is a perfect cube & the exponents on the variables are separated so that one exponent on each variable is divisible by 3.